Math 53: Multivariable Calculus | Sections 102, 103

Handout for 2020-01-31

Summary of last time

I gave the following two questions (the former to my 8AM section, the latter to my 9AM):

Problem. Find the area underneath the parametric curve $x = t^3 + 3t$, $y = 1 - t^2$, $-\infty < t < \infty$ and above the x-axis.

Problem. Find the area enclosed by the innermost loop of the polar curve $r = 2 + \cos(\theta/3)$. (I drew a crude picture of this curve on the board.)

The key conceptual point behind both of these questions is identifying the correct bounds of integration. Their answers are on the back of this sheet of paper.

We discussed how to find intersections and self-intersections. To find the points of intersection of the parametric curves $x = f_1(t)$, $y = g_1(t)$ and $x = f_2(t)$, $y = g_2(t)$, solve the system of equations

$$f_1(u) = f_2(v)$$
$$g_1(u) = g_2(v).$$

The essential point here is that you need to use *different* variables for the two parameters! After all, if we imagine these curves as the trajectory of particles, they could be at the same place at *different* times. (Using the same variable would be appropriate for finding collisions, rather than intersections.)

If we just want to find self-intersections of a parametric curve $x = f(t)$, $y = g(t)$, then we solve

$$f(u) = f(v)$$
$$g(u) = g(v)$$
$$u \neq v.$$

I did the example of $x = t^2 - 1$, $y = t(t^2 - 1)$.

For polar curves, the situation is more complicated. I did the problem from the 1/27 worksheet as an example, but I think that's a bit beyond the difficulty of problems you'll encounter (which is why I put it at the end of that worksheet).

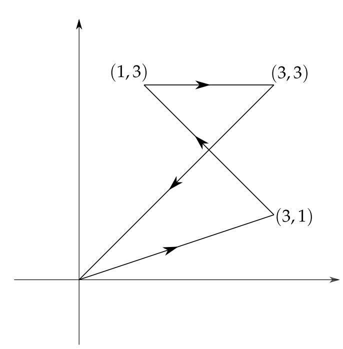
The other thing we discussed was that of putting the correct bounds on the integrals for parametric area. Suppose we have a curve above the x-axis, traversed once either from left to right or from right to left. Let's say the left endpoint is at $t = a$ and $x = x_0$, and the right endpoint at $t = b$ and $x = x_1$. So $x_0 < x_1$, but either of a or b could be the larger ($a < b$ if from left to right, $b < a$ if from right to left). If we want to compute the area underneath this curve and above the x-axis, we use the formula

$$\int_{x_0}^{x_1} y \,\mathrm{d}x = \int_a^b g(t) f'(t) \,\mathrm{d}t$$

where $x = f(t)$ and $y = g(t)$. Note that we put a on the bottom and b on top—even if a happens to be the larger number. Indeed, this is just change of variables from single variable calculus, with the substitution $x = f(t)$. We rewrite $\mathrm{d}x = f'(t)\,\mathrm{d}t$, and we switch the x bounds to the corresponding t bounds. Try the first question below for more about this.

Some questions

Question 1. A particle starts at the origin at time $t = 0$ and follows the path $x = f(t)$, $y = g(t)$ illustrated to the right. At time $t = 1$, it returns to the origin. Compute $\int_0^1 g(t) f'(t)\,\mathrm{d}t$ and $\int_0^1 f(t) g'(t)\,\mathrm{d}t$. How are they related? (Tricky: can you explain why?)



Question 2. Consider the points $A(0, 0, 0)$ and $B(0, 0, 1)$ in space. Fix a positive constant c and consider the set of all points P such that the distance PA is exactly c times the distance PB. What kind of shape does this describe?

Answers to questions from last time

For the first question, use $y = 1 - t^2 \geq 0$ to determine that the correct bounds of integration should be $t = -1, 1$. Which one is the lower bound, and which one is the upper bound? The x value at $t = -1$ is -4 and the x value at $t = 1$ is 4. So the integral to evaluate is $\int_{-1}^{1}(1-t^2)(t^3+3t)\,\mathrm{d}t = \boxed{24/5}$.

For the second question, considering how the curved is traced out shows that the bounds of integration are θ from 2π to 4π. Evaluating $\int_{2\pi}^{4\pi} \frac{r^2}{2}\,\mathrm{d}\theta$, one gets as final answer $\boxed{\frac{9}{2}\pi - \frac{45\sqrt{3}}{8}}$.

Question 1. The answers are −1 and 1 respectively.

They add to zero because the path ends where it starts:

$$\int_0^1 \big(g(t)f'(t) + g'(t)f(t)\big)\,\mathrm{d}t = f(t)g(t)\big|_{t=0}^{1}$$

by the product rule and the FTC.

Question 2. If $c = 1$ then you get a plane, as you saw on a past lecture miniquiz. Otherwise you get a sphere. (Write down the equation and complete the square.)